Problem session

The Problem Session was held on August 5, 1998, in Hódmezővásárhely, chaired by E. T. Schmidt. The following problems were presented.

R. FREESE

PROBLEM 1. Let L be a finitely presented lattice such that every nonzero join irreducible element is completely join irreducible (that is, it has a lower cover) and dually. Is L finite?

We have empirical evidence that this is true and that, when a finitely presented lattice is infinite, there is an element of low rank witnessing this. If these statements could be verified, we would have an alternate way of determining whether a finitely presented lattice is finite.

G. GRÄTZER AND F. WEHRUNG

Let L be a lattice. A lattice K is a *congruence-preserving extension* of L, if K is an extension and every congruence of L has *exactly one* extension to K. Of course, then the congruence lattice of L is isomorphic to the congruence lattice of K; we could say that the congruence lattices are *naturally isomorphic*.

A number of papers have been written on the existence of congruence-preserving extensions with special properties, especially for finite lattices; see, for instance, the references in Appendices A and B in [1].

Here we would like to raise an interesting problem that does not arise for finite lattices:

PROBLEM 2. Let L be a lattice. When does L have a congruence-preserving extension to a lattice with zero?

There are two relevant results, one positive and one negative.

THEROEM A. A distributive lattice L has a congruence-preserving extension to a lattice with zero iff L itself has a zero.

THEOREM B. *Let L be a lattice with a finite congruence lattice. Then L has a congruence-preserving extension to a sectionally complemented lattice.*

The first theorem appears to be new (the proof is easy), while the second was published in G. Grätzer and E. T. Schmidt [2].

Here is an obvious necessary condition for a general lattice L to have a congruence-preserving extension into a lattice with zero. Denote by L^0 the lattice L with a new zero adjoined. If L has a congruence-preserving extension into a lattice with zero, then the natural embedding $j\colon \operatorname{Con} L \hookrightarrow \operatorname{Con} L^0$ admits a *retraction*, that is, a $\bigvee$-complete, $\{\vee, 0\}$-homomorphism $p\colon \operatorname{Con} L^0 \twoheadrightarrow \operatorname{Con} L$ such that $p \circ j$ is the identity. However, note that this condition also follows from the weaker condition that L admits a congruence-preserving extension into a lattice K such that there exists an element of K contained in every element of L.

Different types of problems arise, if we are looking for congruence-preserving extensions into relatively complemented lattices. In the finite case, this is not an interesting question since the congruence lattice of a finite relatively complemented lattice is Boolean. However, as a special case of Theorem B, we obtain that a *finite* lattice always has a congruence-preserving extensions into a (finite) sectionally complemented lattice. On the other hand, M. Ploščica, J. Tůma, and F. Wehrung [4] exhibit a lattice of size $\aleph_2$ that has no congruence-preserving extension into a sectionally complemented lattice.

The following problem is open:

PROBLEM 3. Let L be an infinite lattice with $|L| \leq \aleph_1$. Does L have a congruence-preserving extension to a (sectionally complemented) relatively complemented lattice?

The countable case is known to have a positive answer provided that L is *locally finite*, see J. Tůma [5] and G. Grätzer, H. Lakser, and F. Wehrung [3]. Nothing seems to be known about the case $|L| = \aleph_1$.

PROBLEM 4. Let L be a finite lattice. Does L have a congruence-preserving extension to a (finite) sectionally complemented lattice with the same bounds?

The construction of [2] gives an extension with the same zero, but the unit is not preserved as a rule.

REFERENCES

[1] GRÄTZER, G., *General Lattice Theory. Second Edition*, Birkhäuser Verlag, Basel. 1998. xix+663 pp.

[2] GRÄTZER, G. and SCHMIDT, E. T., *Congruence-preserving extensions of finite lattices into sectionally complemented lattices*, Proc. Amer. Math. Soc., *127* (1999), 1903–1915.

[3] GRÄTZER, G., LAKSER, H. and WEHRUNG, F., *Congruence amalgamation of lattices*, Acta Sci. Math. (Szeged) *66* (2000), 3–22.

[4] PLOŠČICA, M., TŮMA, J. and WEHRUNG, F., *Congruence lattices of free lattices in non-distributive varieties*, Colloq. Math. *76* (1998), 269–278.

[5] TŮMA, J., *On the existence of simultaneous representations*, Acta Sci. Math. (Szeged), *64* (1998), 357–371.

C. HERRMANN

PROBLEM 5. Can every ortholattice be embedded into the principal left ideal lattice of a*-regular ring?

T. KATRIŇAK

Let Fddp(n) denote the free algebra on n generators over the equational class of all distributive double p-algebras.

PROBLEM 6. Determine Fddp(1), or more generally, Fddp(n).

In the paper, *Congruence Lattices of Pseudocomplemented Semilattices*, Semi group Forum **55** (1997), 1–23, I gave a characterization of the congruence lattice Con(S) of an arbitrary pseudocomplemented semilattice S. This description uses a second-order language.

PROBLEM 7. For a finite S, is it possible to find a description of Con(S) in a first-order language?

K. A. KEARNES AND E. W. KISS

PROBLEM 8. Let $A_1, A_2, \ldots, A_n$ be nonempty sets. A rectangular subset of $A_1 \times \cdots \times A_n$ is a nonempty subset of the form $B_1 \times \cdots \times B_n$ with $B_i \subseteq A_i$, for each i. Suppose that $A_1 \times \cdots \times A_n$ is partitioned into fewer than 2^n rectangular subsets. Does it follow that for one of these rectangular subsets $C_1 \times \cdots \times C_n$ there exists an i such that $C_i = A_i$?

PROBLEM 9. Characterize the homomorphic images of strongly Abelian algebras. Is it true that in the case of finite similarity type, they are exactly the strongly nilpotent algebras?

PROBLEM 10. Give a Klukovits type characterization of locally finite weakly Abelian varieties.

PROBLEM 11. Characterize all finite algebras A of finite similarity type such that the number of inequivalent n-ary terms is at most 2^{cn}, for some $c > 0$ and all $n > 0$.

PROBLEM 12. (The restricted Quackenbush Conjecture) Is it true that if V is a finitely generated variety of finite similarity type and all subdirectly irreducibles in V are finite, then there are only finitely many subdirectly irreducibles in V?

PROBLEM 13. (Pixley's Problem) If V is a variety whose subdirectly irreducibles are finite, and whose finitely generated subvarieties are congruence distributive, then must V be congruence distributive?

For background on Problems 8–10, see the paper K. A. Kearnes, E. W. Kiss, *Finite algebras of finite complexity*, Discrete Math. *207* (1999), 89–135.

A. Pinus

An algebra A is called *quasi-simple* if, for any $\Theta \in \mathrm{Con}\, A$ and $\Theta \neq \nabla_A$, there exists $\Theta' \in \mathrm{Con}\, A$ such that $\Theta' > \Theta$ and $A/\Theta' \cong A$.

A lattice L is called *up-indecomposable*, if for any $a \in L$ with $a \neq 1_L$ there exists $a' \in L$ such that $a' > a$ and $F_{a'} \cong L$, where $F_{a'} = \{b \in L, b \geq a'\}$.

PROBLEM 14. For any algebraic up-indecomposable lattice L, does there exist a quasi-simple algebra A such that $\mathrm{Con}\, A \cong L$?

E. T. Schmidt

PROBLEM 15. Let L be a lattice. Does L have a congruence-preserving extension to a lattice with type 3 congruences?

PROBLEM 16. Does every lattice has a congruence-preserving extension to a semimodular lattice?

B. Šešelja and A. Tepavčevič

Let $\mathrm{Conw}A$ denote the lattice of congruences of subalgebras of an algebra A with respect to inclusion. If Δ_A stands for the diagonal relation $\{(a, a) \mid a \in A\}$, then the ideal $(\Delta_A]$ of $\mathrm{Conw}A$ is isomorphic to the subalgebra lattice of A, while the filter $[\Delta_A)$ is the congruence lattice of A.

PROBLEM 17. Let L be an algebraic lattice and let $d \in L$ with $d > 0$. Is there an algebra A and an isomorphism $\psi : L \cong \mathrm{Conw}(L)$ such that ψ sends d to Δ_A?

R. Willard

A decision problem:

PROBLEM 18. Input:

a finite algebra A (in a finite language),
an integer $n > 1$.

Does $V(A)$ contain a SI of cardinality $> n$? Is there an algorithm to decide this?